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Two nice stable maps of C^2P into R^3

by Mahito Kobayashi
Department of Math. Akita Univ., Akita 010, Japan.
(小林 真人 · 秋田大)

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In this short article, we present two stable maps f_1 and $f_2 : \mathbf{C}^2P \to \mathbf{R}^3$ of complex projective space of $\mathbf{C}$-dimension 2 into $\mathbf{R}^3$. They are not only the first explicit examples of stable maps of $\mathbf{C}^2P$ into $\mathbf{R}^3$ but also reflect the geometry of $\mathbf{C}^2P$ naturally. The proofs of the results in this article will be appeared somewhere else.

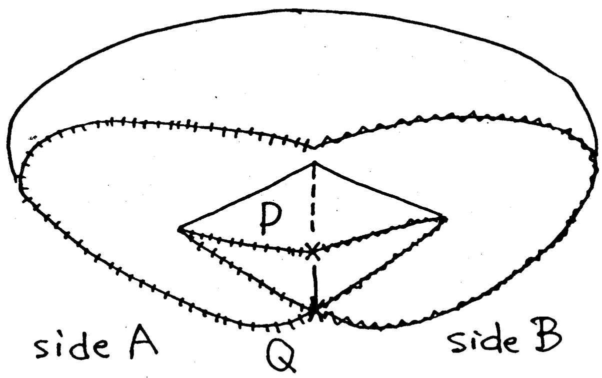
1. The topological aspects of stable maps have been studied by several authors, and some kinds of congruences on the homological data (characteristic classes, etc) of the singular points, the source and the target manifolds have been obtained (see, for example, [**Fuk**],[**Sae**], and [**Sak**]). By using such congruences, Saeki has shown that if a closed 4-manifold M has the homology groups which are isomorphic to that of $\mathbf{C}^2P$, then any stable map $f : M \to \mathbf{R}^3$ has cusp singularities ([**Sae**]). But in contrast to the homological study, the explicit examples of stable maps are scarcely known except for the case where the maps have only fold singularities. No stable maps of even $\mathbf{C}^2P$ into $\mathbf{R}^3$ were known in explicit forms (for the examples of stable maps into $\mathbf{R}^2$, see [**K**]).

Remark that a stable map of a closed 4-manifold into a 3-manifold has the set of singular points which is a smooth submanifold of dimension 2 (or the empty set), and possibly has singular points called swallow tails (of full codimension in the set of singular points), cusps (codimension 1), definite and indefinite folds (codimension 0). The first of our result is

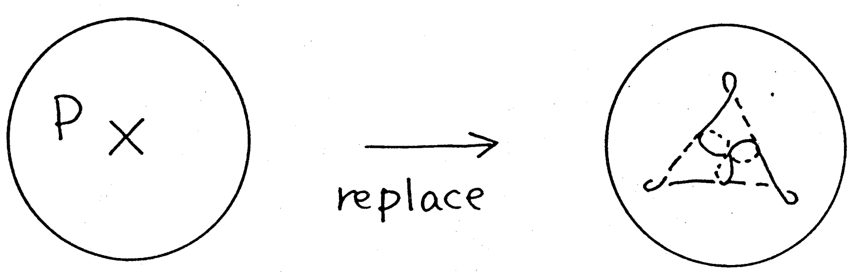
Theorem 0.1 *There exist stable maps $f_i : \mathbf{C}^2P \to \mathbf{R}^3\,(i = 1, 2)$ such that*
1. the set of critical values $C(f_i) \subset \mathbf{R}^3$ is the singular surface in $\mathbf{R}^3$ as described bellow,
2. the map f_1 has 6 swallow tails, 6 arcs of cusps, and some regions of both definite and indefinite folds,
3. the map f_2 has no swallow tails, 1 circle of cusps, and each 1 region of definite and indefinite folds.

(Note that the set of singular points $S(f_1)$ is diffeomorphic to $\mathbf{R}^2P$, and that $S(f_2)$ is diffeomorphic to the disjoint union $S^2 \sqcup \mathbf{R}^2P$; see the figures bellow.)

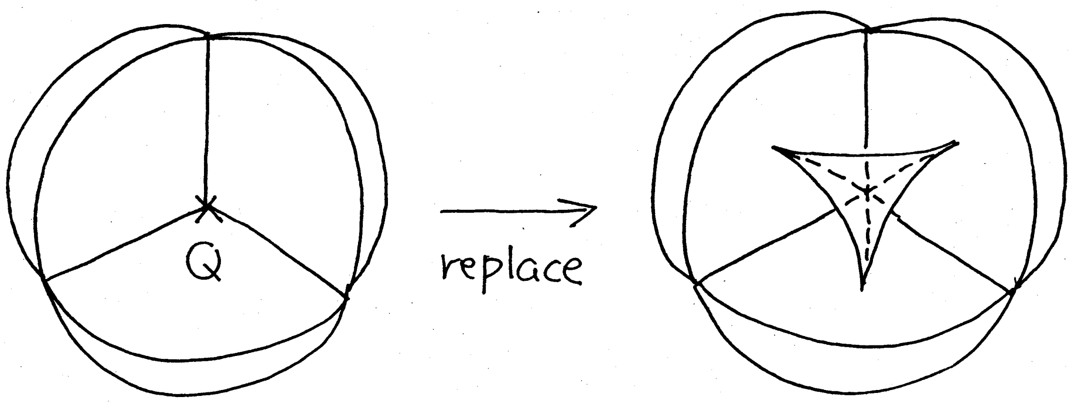
The singular surface $C(f_1)$ is the one obtained in the following manner.
1. Glue three copies of the one below so that side A of a copy is pasted to side B of another.



2. Replace the neighbourhoods of the point P and Q as follows:
At P; replace a disc neighbourhood of P with Boy surface $-$ IntD^2.

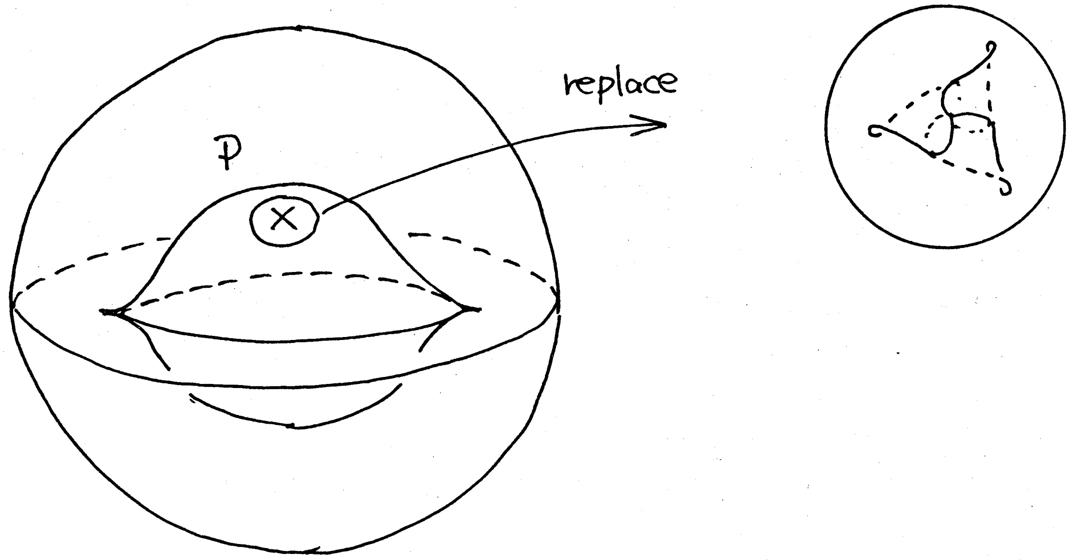


(Note that *Boy surface* is an immersed image of $\mathbf{R}^2P$ into $\mathbf{R}^3$ with three double points loci joined at a unique triple point. Refer to [**Fr**] and the figures in Appendix.) At Q;(refer to the bifercation of wave fronts of D_4^- in [**A**,p 32, Fig. 32]. See also the figure in Appendix).



non-ordinary triple point

The singular surface $C(f_2)$ is the one obtained from "a flying saucer floating in D^3" by replacing a neighbourhood of P with Boy surface $-$ IntD^2 as in $C(f_1)$.



Nice stable maps

The following is a bi-product of Theorem 0.1.

Theorem 0.2 *There exists stable maps g_1 and $g_2 : S^4 \to \mathbf{R}^3$ such that*
1. the set of critical values $C(g_i) \subset \mathbf{R}^3$ is the singular surface obtained by omitting the replacement with Boy surface (the replacement around the point P) in the above procedures of constructing $C(f_i)$,
2. the map g_1 has 3 swallow tails, 3 arcs of cusps and some regions of both definite and indefinite folds,
3. the map g_2 has no swallow tails, 1 circle of cusps and each 1 regions of both definite and indefinite folds.
Note that $S(g_1)$ is diffeomorphic to S^2 and $S(g_2)$ is diffeomorphic to $S^2 \sqcup S^2$.

2. As is well known, $\mathbf{C}^2P$ admits a T^2 action since it is a toric variety. This is, roughly speaking, the action obtained by glueing the three copies of the T^2 action on $\mathbf{C}^2$ defined by

$$(x, y) \mapsto (e^{i\theta}x, e^{i\eta}y).$$

Following Fulton [**Ful**], we write down the quotient map of this action by

$$q : \mathbf{C}^2P \to \mathbf{R}^3, \quad [z_0; z_1; z_2] \mapsto \frac{1}{\Sigma|z_i|^2}(|z_0|^2, |z_1|^2, |z_2|^2).$$

We call q the *moment map* into $\mathbf{R}^3$. Remark that the quotient space of the T^2 action is considered as the triangle located in $\mathbf{R}^3$ with vertices $(1, 0, 0), (0, 1, 0)$ and $(0, 0, 1)$.

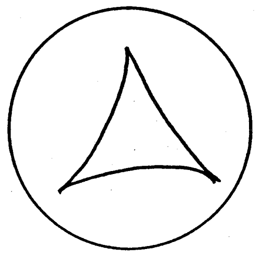
Remark. Our form of the moment map is a slightly different from that stated in [**Ful**]. But the difference can be omitted in the smooth category, *i.e.*, these two moment maps are smoothly equivalent.

Theorem 0.3 *Both of the smooth maps f_1 and f_2 in Theorem 0.1 are the stable perturbations of the moment map q.*

Let $\gamma : \mathbf{R}^3 \to \mathbf{R}^2$ be the linear projection onto the plane which contains the triangle $q(\mathbf{C}^2P)$. We call $\gamma \circ q : \mathbf{C}^2P \to \mathbf{R}^2$ the moment map into $\mathbf{R}^2$.

Nice stable maps

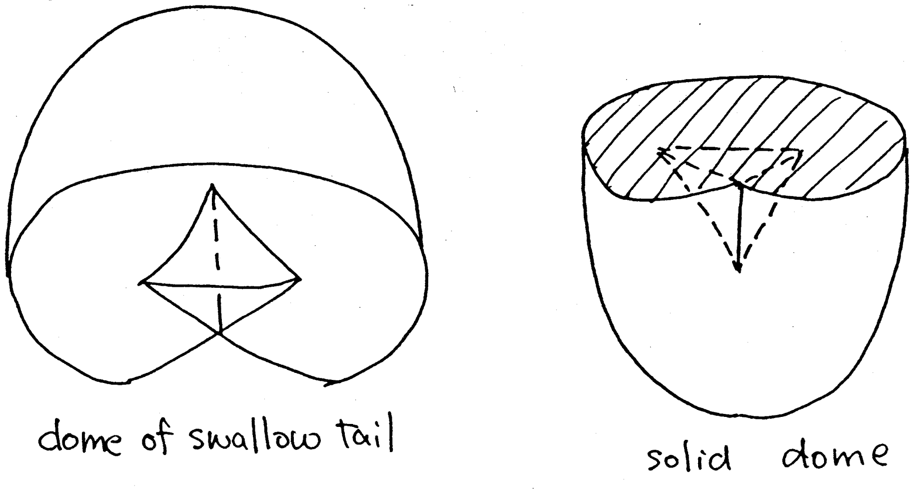
Corollary 0.4 *Let f_1 and f_2 be as before. Then*
1. $\gamma \circ f_i : \mathbf{C}^2P \to \mathbf{R}^2\,(i = 1, 2)$ is a stable map such that $\gamma \circ f_1 = \gamma \circ f_2$.
2. The set of critical values $C(\gamma \circ f_i)$ is as illustrated in the figure. (Note that the set of singular points $S(\gamma \circ f_i)$ is the disjoint union of two circles, which contains 3 cusps, 3 arcs of indefinite folds, and 1 circle of definite folds.)
3. $\gamma \circ f_i$ is a stable perturbation of the moment map $\gamma \circ q$ into $\mathbf{R}^2$.



Remark. The map $\gamma \circ f_i$ is the one constructed in [K, p343].

3. In place of describing the proofs of the above results, we expose a key proposition of the construction of f_1. The map f_1 is obtained by glueing the three copies of $f_{1/3}$ in the proposition after moving them so that they form a map of $\mathbf{C}^2P$.

Let Σ be the singular surface in $\mathbf{R}^3$ which is obtained from the critical locus near a swallow tail by stretching the "base sheet" so that it covers the swallow tail singularity (see the figure). We call Σ the dome of swallow tail. Let L be the plane which contains the boundary of the singular surface Σ. Note that Σ together with L bounds a 3-space. We call it the solid dome of swallow tail.



Proposition 0.5 *There exists a smooth map $f_{1/3} : D^4 \to \mathbf{R}^3$ such that*
1. the image is the solid dome of swallow tail,
2. $f_{1/3}$ maps ∂D^4 into the plane L which contains $\partial\Sigma$, and $f_{1/3}|\partial D^4$ is a stable map into L,
3. on $\mathrm{Int}D^4$, $f_{1/3}$ is a stable map (with 1 swallow tail, 2 arcs of cusps, both 1 regions of definite and indefinite folds),
4. the set of critical values $C(f_{1/3})$ (which is the union of $C(f_{1/3}|\partial D^4)$ and $C(f_{1/3}|\mathrm{Int}D^4)$) is the dome of swallow tail.

We remark that on any nonsingular toric surfaces, one can construct stable maps which are the perturbations of the moment maps by using the same method as the proofs of Theorem 0.1 and 0.3.

It is a pleasure to thank Toshizumi Fukui for helpful conversations. The finding of the map f_2 and the names "dome" and "flying saucer" due to him.

References

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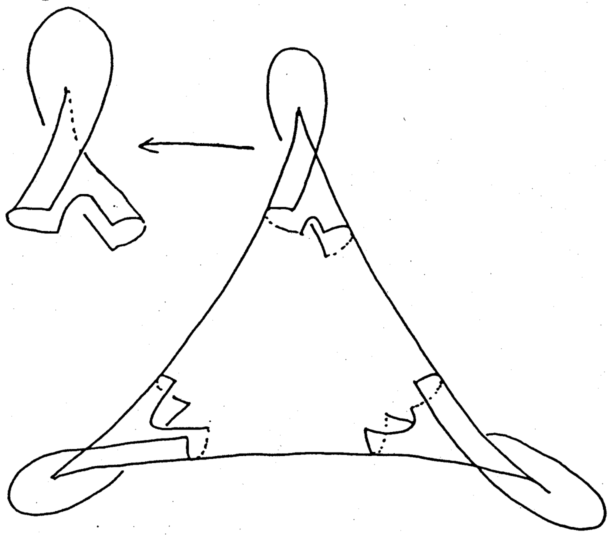
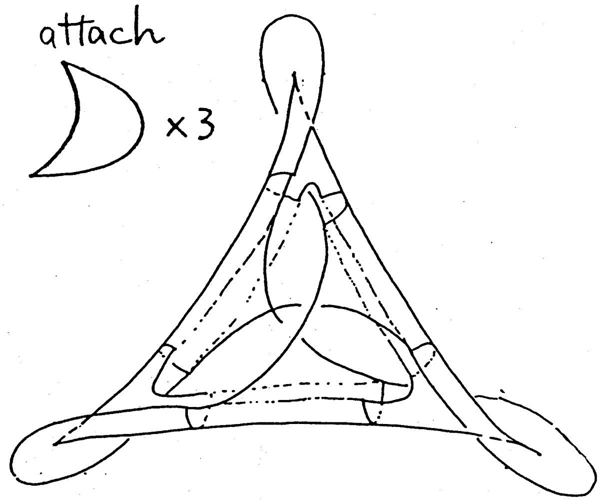
Appendix

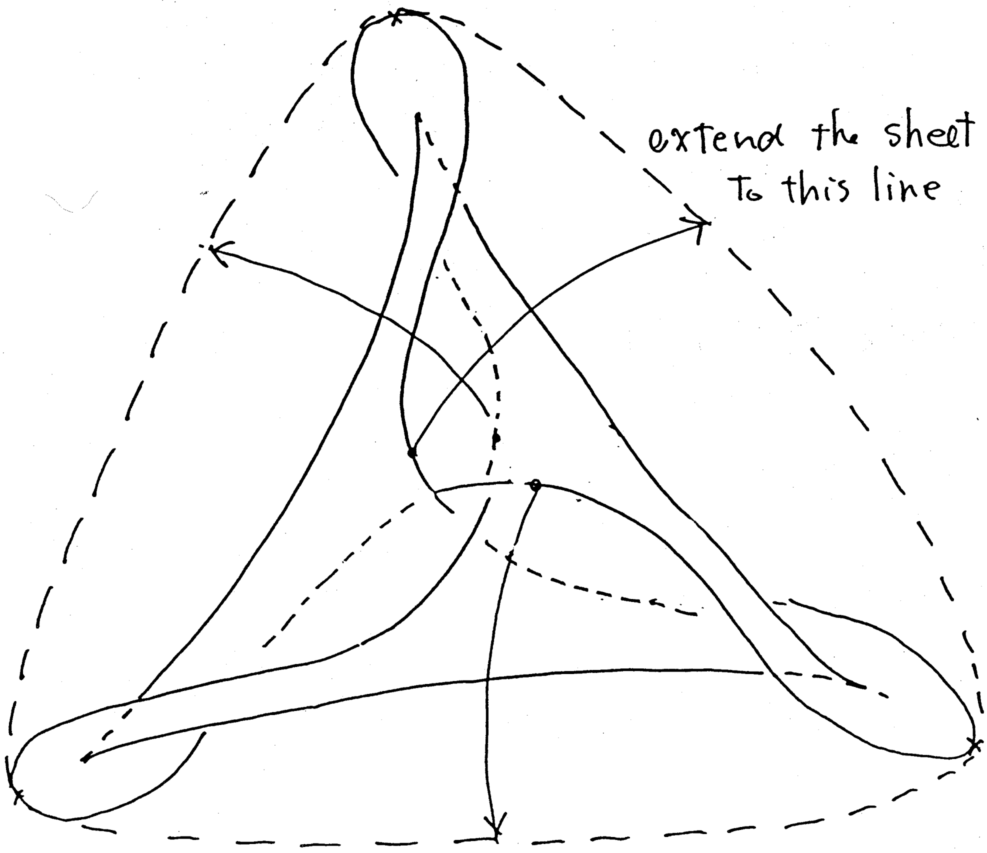
Figures on constructing Boy surface$-\mathrm{Int}D^2$



tie the 3 cusps with fold arcs



enlarge



Boy surface $-$ Int D^2

The 1/3 piece of the renewed neighbourhood of the non-ordinary triple point Q